Taylor and Maclaurin Series

Infinite Sequences and Series · 6 questions · step-by-step answers at the end

Name: Date: Score: / 6

USEFUL EQUATIONS

$f(x) = \Sigma\, f^{(n)}(a)(x-a)^n/n!$ $\quad e^x = 1 + x + x^2/2! + x^3/3! + \ldots$

$\sin x = x - x^3/3! + x^5/5! - \ldots$, $\cos x = 1 - x^2/2! + x^4/4! - \ldots$

$1/(1-x) = 1 + x + x^2 + x^3 + \ldots$

Questions

Show your working in the space under each question.

1. EASY Write the Maclaurin polynomial of degree 4 for $\cos x$ and estimate $\cos 0.2$.

2. EASY Find the first three non-zero terms of the Maclaurin series for e^{-x^2}.

3. MEDIUM Find the degree-2 Taylor polynomial for $\sqrt{x}$ about $x = 4$.

4. MEDIUM Find the radius of convergence of $\Sigma\, x^n/n!$ and of $\Sigma\, nx^n$.

5. HARD Bound the error when sin x is approximated by $x - x^3/6$ for $|x| \le 0.5$.

6. HARD Find the interval of convergence of $\Sigma\ (x + 1)^n/(n^2 \cdot 2^n)$, $n \ge 1$.

Step-by-step answers

Taylor and Maclaurin Series

1. $P_4 = 1 - x^2/2 + x^4/24$. $P_4(0.2) = 1 - 0.02 + 0.0000667 =$ **0.980067**.

2. Substitute $-x^2$ into $e^u = 1 + u + u^2/2$: $\mathbf{1 - x^2 + x^4/2}$.

3. $f(4) = 2$. $f' = 1/(2\sqrt{x}) = 1/4$. $f'' = -(1/4)x^{-3/2} = -1/32$, and dividing by 2! gives $-1/64$. $\mathbf{P_2 = 2 + (x - 4)/4 - (x - 4)^2/64}$.

4. First: ratio $|x|/(n + 1) \to 0$ for every x, so $\mathbf{R = \infty}$. Second: ratio $|x|(n + 1)/n \to |x|$, so $\mathbf{R = 1}$.

5. The series alternates, and the next term is $x^5/120$. Error $\le 0.5^5/120 = 0.03125/120 \approx$ **0.00026**.

6. Ratio $\to |x + 1|/2 < 1$ gives $-3 < x < 1$. At both endpoints the terms have size $1/n^2$, a convergent p-series. Interval: $\mathbf{-3 \le x \le 1}$.